Separability criteria for several classes of n-partite quantum states

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In this paper, we mainly discuss the separability of n-partite quantum states from elements of density matrices. Practical separability criteria for different classes of n-qubit and n-qudit quantum states are obtained. Some of them are also sufficient conditions for genuine entanglement of n-partite quantum states. Moreover, one of the resulting criteria is also necessary and sufficient for a class of n-partite states.

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I. INTRODUCTION

Quantum entanglement is a kind of new resources beyond the classical resources, and has widely been applied to quantum communication [1–6] and quantum computation [7, 8]. Whether a state is entangled or not is one of the most challenging open problems. For the states of 2×2 and 2×3 bipartite systems, they are separable iff they are positive partial transposition (PPT) [9, 10]. For high dimensional and multipartite systems, however, the situation is significantly more complicated, as several inequivalent classes of multiparticle entanglement exist and it is difficult to decide to which class a given state belongs.

It would be desirable to have useful criteria that allow us to detect the different classes of multipartite entanglement directly from a given density matrix. Gühne and Seevinck [11] presented a method to derive separability criteria for different classes of 3-qubit and 4-qubit entanglement, especially genuine 3-qubit and 4-qubit entanglement. Huber et al. [12] developed a general framework to identify genuinely multipartite entangled mixed quantum states in arbitrary-dimensional systems. Based on the framework, k-separability criterion was derived in [13].

In this paper, the separability of n-partite and multilevel quantum states from elements of density matrices is investigated. We derive simple algebraic tests, which are necessary conditions for separability of n-partite quantum states. Some of them are also sufficient conditions for genuine entanglement of n-qubit and n-qudit quantum states. One of the resulting criteria is necessary and sufficient for a certain family of n-partite states.

An n-partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \mathcal{H}_n$ is called biseparable if there is a bipartition $j_1 j_2 \cdots j_k | j_{k+1} \cdots j_n$ such that

$$|\psi\rangle = |\psi_1\rangle_{j_1 j_2 \cdots j_k} |\psi_2\rangle_{j_{k+1} \cdots j_n}, \tag{1}$$

where $|\psi_1\rangle_{j_1 j_2 \cdots j_k}$ is the state of particles $j_1, j_2, \cdots, j_k$, $|\psi_2\rangle_{j_{k+1} \cdots j_n}$ is the state of particles $j_{k+1}, \cdots, j_n$, and $\{j_1, j_2, \cdots, j_n\} = \{1, 2, \cdots, n\}$. An n-partite mixed state ρ is biseparable if it can be written as a convex combination of biseparable pure states

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{2}$$

where $|\psi_i\rangle$ might be biseparable under different partitions. If an n-partite state is not biseparable, then it is called genuinely n-partite entangled. Genuine n-partite entanglement is very important as one usually aims to generate and verify this class of entanglement in experiments [14]. We mainly discuss entanglement criteria for this type of entanglement. An n-partite pure state is fully separable if it is of the form

$$|\psi\rangle = |\psi\rangle_1 |\psi\rangle_2 \cdots |\psi\rangle_n, \tag{3}$$

and an n-partite mixed state is fully separable if it is a mixture of fully separable pure states

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{4}$$

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where the p_i forms a probability distribution, and $|\psi_i\rangle$ is fully separable. We also consider separability criteria of biseparable and fully separable n-qubit and n-qudit states, and give clear and complete proof of each criterion from general partition by using the Cauchy inequality and Hölder inequality.

II. THE SEPARABILITY CRITERIA OF BISEPARABLE n-PARTITE STATES AND GENUINE n-PARTITE ENTANGLED STATES

Let ρ be a density matrix describing an n-particle system, whose state space is Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \mathcal{H}_n$, where $\dim\mathcal{H}_l = d_l$, $l = 1, 2, \cdots, n$. We denote its entries by $\rho_{i,j}$, where $1 \leq i, j \leq d_1 d_2 \cdots d_n$.

Next we investigate biseparable n-partite states and genuine n-partite entangled states.

Theorem 1 (Gühne and Seevinck [11]) For any n-qubit density matrix, $\rho = (\rho_{i,j})_{2^n \times 2^n}$, if it is biseparable, then

$$|\rho_{1,2^n}| \leq \sum_{i=2}^{2^{n-1}} \sqrt{\rho_{i,i}\rho_{2^n-i+1,2^n-i+1}} = \frac{1}{2}\sum_{i=2}^{2^n-1} \sqrt{\rho_{i,i}\rho_{2^n-i+1,2^n-i+1}}. \tag{5}$$

That is, if the inequality (5) does not hold, then ρ is a genuine n-qubit entangled state.

Proof. First we show that (5) holds for pure state.

Suppose that $\rho = |\psi\rangle\langle\psi|$ is an n-qubit pure biseparable state under the $j_1 j_2 \cdots j_k | j_{k+1} \cdots j_n$ partition, and

$$\begin{aligned} |\psi\rangle = & \ |\phi_1\rangle_{j_1 j_2 \cdots j_k} |\phi_2\rangle_{j_{k+1} \cdots j_n} \\ = & \ (\sum_{i_1, i_2, \cdots, i_k = 0}^{1} a_{i_1 i_2 \cdots i_k} |i_1 i_2 \cdots i_k\rangle)_{j_1 j_2 \cdots j_k} (\sum_{i_{k+1}, \cdots, i_n = 0}^{1} b_{i_{k+1} \cdots i_n} |i_{k+1} \cdots i_n\rangle)_{j_{k+1} \cdots j_n}, \end{aligned} \tag{6}$$

then

$$\rho = |\psi\rangle\langle\psi| = \sum_{\substack{i_1, i_2, \cdots, i_n \\ i'_1, i'_2, \cdots, i'_n}} a_{i_1 i_2 \cdots i_k} b_{i_{k+1} \cdots i_n} a^*_{i'_1 i'_2 \cdots i'_k} b^*_{i'_{k+1} \cdots i'_n} |i_1 i_2 \cdots i_n\rangle_{j_1 j_2 \cdots j_n} \langle i'_1 i'_2 \cdots i'_n|, \tag{7}$$

where $\{j_1, j_2, \cdots, j_n\} = \{1, 2, \cdots, n\}$. From

$$\begin{aligned} \rho_{1,2^n} &= a_{00\cdots 0} b_{00\cdots 0} a^*_{11\cdots 1} b^*_{11\cdots 1}, \\ \rho_{\sum_{l=1}^{k} 2^{n-j_l}+1, \sum_{l=1}^{k} 2^{n-j_l}+1} &= |a_{11\cdots 1} b_{00\cdots 0}|^2, \\ \rho_{\sum_{l=k+1}^{n} 2^{n-j_l}+1, \sum_{l=k+1}^{n} 2^{n-j_l}+1} &= |a_{00\cdots 0} b_{11\cdots 1}|^2, \end{aligned} \tag{8}$$

one has

$$|\rho_{1,2^n}| = \sqrt{\rho_{\sum_{l=1}^{k} 2^{n-j_l}+1, \sum_{l=1}^{k} 2^{n-j_l}+1} \rho_{\sum_{l=k+1}^{n} 2^{n-j_l}+1, \sum_{l=k+1}^{n} 2^{n-j_l}+1}}. \tag{9}$$

Clearly, $\sum_{l=1}^{k} 2^{n-j_l} + 1 = 2, 3, \cdots, 2^n - 1$ for $\{j_1, j_2, \cdots, j_n\} = \{1, 2, \cdots, n\}$. Thus, (5) holds for pure state ρ.

Next we prove that the inequality (5) is also right for mixed states.

Suppose that

$$\rho = \sum_i p_i \rho^{(i)} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{10}$$

is biseparable n-qubit state, where $\rho^{(i)} = |\psi_i\rangle\langle\psi_i|$ is biseparable. Simple algebra and the Cauchy inequality $(\sum_{k=1}^{m} x_k y_k)^2 \leq (\sum_{k=1}^{m} x_k^2)(\sum_{k=1}^{m} y_k^2)$ show that

$$\begin{aligned} |\rho_{1,2^n}| = & \ |\sum_i p_i \rho^{(i)}_{1,2^n}| \leq \sum_i p_i |\rho^{(i)}_{1,2^n}| \\ \leq & \ \sum_i p_i \sum_{j=2}^{2^{n-1}} \sqrt{\rho^{(i)}_{j,j} \rho^{(i)}_{2^n-j+1,2^n-j+1}} \\ \leq & \ \sum_{j=2}^{2^{n-1}} \sqrt{(\sum_i p_i \rho^{(i)}_{j,j})(\sum_i p_i \rho^{(i)}_{2^n-j+1,2^n-j+1})} \\ = & \ \sum_{j=2}^{2^{n-1}} \sqrt{\rho_{j,j} \rho_{2^n-j+1,2^n-j+1}}. \end{aligned} \tag{11}$$

The proof is complete.

The same result in this theorem has also been derived in [11]. Gühne and Seevinck [11] proved the cases of $n=3,4$. Here starting from general bipartition for n-qubit pure states and applying the Cauchy inequality, we give a proof for any n-qubit states.

Moreover, for n-partite and high dimension system, we have:

Theorem 2 Suppose that n-partite density matrix $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \mathcal{H}_n$, $\dim\mathcal{H}_l = d_l$, $l=1,2,\cdots,n$. If ρ is biseparable, then

$$|\rho_{1,d_1d_2\cdots d_n}| \leq \frac{1}{2}\sum_{i\in A}\sqrt{\rho_{i,i}\rho_{d_1d_2\cdots d_n-i+1,d_1d_2\cdots d_n-i+1}}, \tag{12}$$

where $A=\{\sum_{l=1}^{n-1} i_l d_{l+1}\cdots d_n + i_n + 1 \mid i_l = 0, d_l-1, (i_1,i_2,\cdots,i_n)\neq(0,0,\cdots,0),(d_1-1,d_2-1,\cdots,d_n-1)\}$. Of course, ρ is a genuine n-partite entangled state if it violates the above inequality (12).

Proof. Suppose that $\rho=|\psi\rangle\langle\psi|$ is a biseparable pure state under the $j_1j_2\cdots j_k|j_{k+1}\cdots j_n$ partition, and

$$\begin{aligned}|\psi\rangle &= |\psi_1\rangle_{j_1j_2\cdots j_k}|\psi_2\rangle_{j_{k+1}\cdots j_n}\\ &= (\sum_{i_1,i_2,\cdots,i_k} a_{i_1i_2\cdots i_k}|i_1i_2\cdots i_k\rangle)_{j_1j_2\cdots j_k}(\sum_{i_{k+1},\cdots,i_n} b_{i_{k+1}\cdots i_n}|i_{k+1}\cdots i_n\rangle)_{j_{k+1}\cdots j_n}\\ &= \sum_{i_1,i_2,\cdots,i_n} a_{i_1i_2\cdots i_k}b_{i_{k+1}\cdots i_n}|i_1i_2\cdots i_n\rangle_{j_1j_2\cdots j_n},\end{aligned} \tag{13}$$

then

$$\rho_{\sum\limits_{l=1}^{n} i_l d_{j_l+1}d_{j_l+2}\cdots d_n d_{n+1}+1,\sum\limits_{l=1}^{n} i'_l d_{j_l+1}d_{j_l+2}\cdots d_n d_{n+1}+1} = a_{i_1i_2\cdots i_k}b_{i_{k+1}\cdots i_n}a^*_{i'_1i'_2\cdots i'_k}b^*_{i'_{k+1}\cdots i'_n}. \tag{14}$$

Here the sum is over all possible values of $i_1,i_2,\cdots,i_n$, i.e., $\sum_{i_1,i_2,\cdots,i_n} = \sum_{i_1=0}^{d_{j_1}-1}\sum_{i_2=0}^{d_{j_2}-1}\cdots\sum_{i_n=0}^{d_{j_n}-1}$, $d_{n+1}=1$, and $\{j_1,j_2,\cdots,j_n\}=\{1,2,\cdots,n\}$.

Since

$$\begin{aligned}\rho_{1,d_1d_2\cdots d_n} &= a_{00\cdots0}b_{00\cdots0}a^*_{d_{j_1}-1d_{j_2}-1\cdots d_{j_k}-1}b^*_{d_{j_{k+1}}-1d_{j_{k+2}}-1\cdots d_{j_n}-1},\\ \rho_{\sum\limits_{l=1}^{k}(d_{j_l}-1)d_{j_l+1}d_{j_l+2}\cdots d_nd_{n+1}+1,\sum\limits_{l=1}^{k}(d_{j_l}-1)d_{j_l+1}d_{j_l+2}\cdots d_nd_{n+1}+1} &= |a_{d_{j_1}-1d_{j_2}-1\cdots d_{j_k}-1}b_{00\cdots0}|^2,\\ \rho_{\sum\limits_{l=k+1}^{n}(d_{j_l}-1)d_{j_l+1}d_{j_l+2}\cdots d_nd_{n+1}+1,\sum\limits_{l=k+1}^{n}(d_{j_l}-1)d_{j_l+1}d_{j_l+2}\cdots d_nd_{n+1}+1} &= |a_{00\cdots0}b_{d_{j_{k+1}}-1d_{j_{k+2}}-1\cdots d_{j_n}-1}|^2,\end{aligned} \tag{15}$$

these give

$$\begin{aligned}|\rho_{1,d_1d_2\cdots d_n}| = &\sqrt{\rho_{\sum\limits_{l=1}^{k}(d_{j_l}-1)d_{j_l+1}d_{j_l+2}\cdots d_nd_{n+1}+1,\sum\limits_{l=1}^{k}(d_{j_l}-1)d_{j_l+1}d_{j_l+2}\cdots d_nd_{n+1}+1}}\\ &\times\sqrt{\rho_{\sum\limits_{l=k+1}^{n}(d_{j_l}-1)d_{j_l+1}d_{j_l+2}\cdots d_nd_{n+1}+1,\sum\limits_{l=k+1}^{n}(d_{j_l}-1)d_{j_l+1}d_{j_l+2}\cdots d_nd_{n+1}+1}}.\end{aligned} \tag{16}$$

Thus, (12) holds for pure state ρ.

Next we prove that the inequality (12) is also right for mixed states.

Suppose that

$$\rho = \sum_i p_i\rho^{(i)} = \sum_i p_i|\psi_i\rangle\langle\psi_i| \tag{17}$$

is a biseparable n-partite mixed state, where $\rho^{(i)}=|\psi_i\rangle\langle\psi_i|$ is biseparable. With the help of (12) for pure states $\rho^{(i)}$ and the Cauchy inequality $(\sum\limits_{k=1}^{m} x_ky_k)^2 \leq (\sum\limits_{k=1}^{m} x_k^2)(\sum\limits_{k=1}^{m} y_k^2)$, there is

$$\begin{aligned}|\rho_{1,d_1d_2\cdots d_n}| =& \ |\sum_i p_i\rho^{(i)}_{1,d_1d_2\cdots d_n}| \leq \sum_i p_i|\rho^{(i)}_{1,d_1d_2\cdots d_n}|\\ \leq& \ \sum_i p_i\left(\frac{1}{2}\sum_{j\in A}\sqrt{\rho^{(i)}_{j,j}\rho^{(i)}_{d_1d_2\cdots d_n-j+1,d_1d_2\cdots d_n-j+1}}\right)\\ \leq& \ \frac{1}{2}\sum_{j\in A}\sqrt{(\sum_i p_i\rho^{(i)}_{j,j})(\sum_i p_i\rho^{(i)}_{d_1d_2\cdots d_n-j+1,d_1d_2\cdots d_n-j+1})}\\ =& \ \frac{1}{2}\sum_{j\in A}\sqrt{\rho_{j,j}\rho_{d_1d_2\cdots d_n-j+1,d_1d_2\cdots d_n-j+1}},\end{aligned} \tag{18}$$

as required.

Ineqs. (5) and (12) can also be obtained from inequality (II) in Ref.[12] when $|\Phi\rangle = |00\cdots0\rangle|11\cdots1\rangle$ and $|\Phi\rangle = |00\cdots0\rangle|(d_1-1)(d_2-1)\cdots(d_n-1)\rangle$, respectively. Here we give different proofs.

For n-qubit states, there is:

Theorem 3 Let ρ be an n-qubit state. If ρ is biseparable, then its matrix entries fulfill

$$\sum_{0\leq i<j\leq n-1} |\rho_{2^i+1,2^j+1}| \leq \sum_{0\leq i<j\leq n-1} \sqrt{\rho_{1,1}\rho_{2^i+2^j+1,2^i+2^j+1}} + \frac{n-2}{2}\sum_{i=0}^{n-1}\rho_{2^i+1,2^i+1}, \tag{19}$$

i.e.,

$$\sum_{1\leq j<i\leq n} |\rho_{2^{n-i}+1,2^{n-j}+1}| \leq \sum_{1\leq j<i\leq n} \sqrt{\rho_{1,1}\rho_{2^{n-i}+2^{n-j}+1,2^{n-i}+2^{n-j}+1}} + \frac{n-2}{2}\sum_{i=1}^{n}\rho_{2^{n-i}+1,2^{n-i}+1}. \tag{20}$$

If n-qubit state ρ does not satisfy the above inequality (19) or (20), then ρ is genuine n-partite entangled.

Proof. We begin with pure state. Suppose that $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = |\phi_1\rangle_{m_1m_2\cdots m_k}|\phi_2\rangle_{m_{k+1}\cdots m_n}$, $\{m_1, m_2, \cdots, m_n\} = \{1, 2, \cdots, n\}$. For any $1 \leq j < i \leq n$, it is not difficult to prove that

$$\begin{aligned} |\rho_{2^{n-i}+1,2^{n-j}+1}| &= \sqrt{\rho_{2^{n-i}+1,2^{n-i}+1}\rho_{2^{n-j}+1,2^{n-j}+1}} \\ &\leq \tfrac{\rho_{2^{n-i}+1,2^{n-i}+1}+\rho_{2^{n-j}+1,2^{n-j}+1}}{2} \end{aligned} \tag{21}$$

in the case either $i, j \in A$ or $i, j \in B$, and

$$|\rho_{2^{n-i}+1,2^{n-j}+1}| = \sqrt{\rho_{1,1}\rho_{2^{n-i}+2^{n-j}+1,2^{n-i}+2^{n-j}+1}} \tag{22}$$

in the case one of i and j in A while another in B (either $i \in A$, $j \in B$, or $i \in B$, $j \in A$). Here $A = \{m_1, m_2, \cdots, m_k\}$ and $B = \{m_{k+1}, m_{k+2}\cdots, m_n\}$. Combining (21) and (22) gives that

$$\begin{aligned} &\sum_{1\leqslant j<i\leqslant n} |\rho_{2^{n-i}+1,2^{n-j}+1}| \\ =\ &\sum_{\substack{1\leqslant j<i\leqslant n\\ i\in A, j\in B}} |\rho_{2^{n-i}+1,2^{n-j}+1}| + \sum_{\substack{1\leqslant j<i\leqslant n\\ j\in A, i\in B}} |\rho_{2^{n-i}+1,2^{n-j}+1}| + \sum_{\substack{1\leqslant j<i\leqslant n\\ i,j\in A}} |\rho_{2^{n-i}+1,2^{n-j}+1}| + \sum_{\substack{1\leqslant j<i\leqslant n\\ i,j\in B}} |\rho_{2^{n-i}+1,2^{n-j}+1}| \\ \leq\ &\sum_{\substack{1\leqslant j<i\leqslant n\\ i\in A, j\in B}} \sqrt{\rho_{1,1}\rho_{2^{n-i}+2^{n-j}+1,2^{n-i}+2^{n-j}+1}} + \sum_{\substack{1\leqslant j<i\leqslant n\\ j\in A, i\in B}} \sqrt{\rho_{1,1}\rho_{2^{n-i}+2^{n-j}+1,2^{n-i}+2^{n-j}+1}} \\ &+ \sum_{\substack{1\leqslant j<i\leqslant n\\ i,j\in A}} \tfrac{\rho_{2^{n-i}+1,2^{n-i}+1}+\rho_{2^{n-j}+1,2^{n-j}+1}}{2} + \sum_{\substack{1\leqslant j<i\leqslant n\\ i,j\in B}} \tfrac{\rho_{2^{n-i}+1,2^{n-i}+1}+\rho_{2^{n-j}+1,2^{n-j}+1}}{2} \\ \leq\ &\sum_{1\leq j<i\leq n} \sqrt{\rho_{1,1}\rho_{2^{n-i}+2^{n-j}+1,2^{n-i}+2^{n-j}+1}} + \tfrac{n-2}{2}\sum_{i=1}^{n}\rho_{2^{n-i}+1,2^{n-i}+1}. \end{aligned} \tag{23}$$

that is, (20) holds for any biseparable n-qubit pure state ρ.

Now we suppose that $\rho = \sum_m p_m\rho^{(m)}$ is a biseparable mixed state, and $\rho^{(m)} = |\psi_m\rangle\langle\psi_m|$ is biseparable. Then, simple algebra and the Cauchy inequality show that

$$\begin{aligned} &\sum_{1\leqslant j<i\leqslant n} |\rho_{2^{n-i}+1,2^{n-j}+1}| \\ =\ &\sum_{1\leqslant j<i\leqslant n} |\sum_m p_m\rho^{(m)}_{2^{n-i}+1,2^{n-j}+1}| \\ \leq\ &\sum_m p_m \sum_{1\leqslant j<i\leqslant n} |\rho^{(m)}_{2^{n-i}+1,2^{n-j}+1}| \\ \leq\ &\sum_m p_m \left(\sum_{1\leq j<i\leq n} \sqrt{\rho^{(m)}_{1,1}\rho^{(m)}_{2^{n-i}+2^{n-j}+1,2^{n-i}+2^{n-j}+1}} + \tfrac{n-2}{2}\sum_{i=1}^{n}\rho^{(m)}_{2^{n-i}+1,2^{n-i}+1}\right) \\ =\ &\sum_{1\leq j<i\leq n}\sum_m \sqrt{p_m\rho^{(m)}_{1,1}}\sqrt{p_m\rho^{(m)}_{2^{n-i}+2^{n-j}+1,2^{n-i}+2^{n-j}+1}} + \tfrac{n-2}{2}\sum_{i=1}^{n}\sum_m p_m\rho^{(m)}_{2^{n-i}+1,2^{n-i}+1} \\ \leq\ &\sum_{1\leq j<i\leq n} \sqrt{\sum_m p_m\rho^{(m)}_{1,1}}\sqrt{\sum_m p_m\rho^{(m)}_{2^{n-i}+2^{n-j}+1,2^{n-i}+2^{n-j}+1}} + \tfrac{n-2}{2}\sum_{i=1}^{n}\sum_m p_m\rho^{(m)}_{2^{n-i}+1,2^{n-i}+1} \\ =\ &\sum_{1\leq j<i\leq n} \sqrt{\rho_{1,1}\rho_{2^{n-i}+2^{n-j}+1,2^{n-i}+2^{n-j}+1}} + \tfrac{n-2}{2}\sum_{i=1}^{n}\rho_{2^{n-i}+1,2^{n-i}+1}, \end{aligned} \tag{24}$$

which is the desired result.

Observation 3 and Observation 4 (ii) in [11] are the special cases $n = 3$ and $n = 4$ of Theorem 3, respectively.

III. THE SEPARABILITY CRITERIA OF FULLY SEPARABLE n-PARTITE STATES

In this section, we consider fully separable n-partite states.

For fully separable n-qubit states, by utilizing the Cauchy inequality and Hölder inequality, we derive:

Theorem 4 If an n-qubit density matrix ρ is fully separable, then the following inequalities hold:

$$|\rho_{1,2^n}| \leq (\rho_{2,2}\rho_{3,3}\rho_{4,4}\cdots\rho_{2^n-1,2^n-1})^{\frac{1}{2^n-2}}, \tag{25}$$

$$\sum_{0\leq i<j\leq n-1} |\rho_{2^i+1,2^j+1}| \leq \sum_{0\leq i<j\leq n-1} \sqrt{\rho_{1,1}\rho_{2^i+2^j+1,2^i+2^j+1}}. \tag{26}$$

These two inequalities are equalities for fully separable n-partite pure states.

Proof. First, let us start with pure states.

Suppose that $\rho = |\psi\rangle\langle\psi|$ is a fully sepaprable n-qubit pure state, where

$$\begin{aligned} |\psi\rangle &= (a_{10}|0\rangle + a_{11}|1\rangle)\otimes(a_{20}|0\rangle + a_{21}|1\rangle)\otimes\cdots\otimes(a_{n0}|0\rangle + a_{n1}|1\rangle) \\ &= \sum_{i_1,\cdots,i_n=0}^{1} a_{1i_1}a_{2i_2}\cdots a_{ni_n}|i_1 i_2\cdots i_n\rangle. \end{aligned} \tag{27}$$

Then

$$\rho_{i,j} = a_{1i_1}a_{2i_2}\cdots a_{ni_n}a^*_{1j_1}a^*_{2j_2}\cdots a^*_{nj_n}, \tag{28}$$

where $i = \sum_{k=1}^n i_k\cdot 2^{n-k}+1$, $j = \sum_{k=1}^n j_k\cdot 2^{n-k}+1$. It follows that

$$\begin{aligned} &\rho_{2,2}\rho_{3,3}\cdots\rho_{2^n-1,2^n-1} \\ =\ & |a_{10}a_{20}\cdots a_{n-10}a_{n1}|^2|a_{10}a_{20}\cdots a_{n-11}a_{n0}|^2\cdots|a_{11}a_{21}\cdots a_{n-11}a_{n0}|^2 \\ =\ & |a_{10}a_{20}\cdots a_{n0}a_{11}a_{21}\cdots a_{n1}|^{2^n-2} \\ =\ & (\rho_{1,2^n})^{2^n-2}, \end{aligned} \tag{29}$$

that is, the inequality (25) is an equality for fully separable n-qubit pure states.

Note that

$$\begin{aligned} &\rho_{\sum_{l=1}^t 2^{n-k_l}+1,\sum_{l=1}^t 2^{n-k_l}+1}\rho_{\sum_{l=t+1}^n 2^{n-k_l}+1,\sum_{l=t+1}^n 2^{n-k_l}+1} \\ =\ & |a_{k_1 1}a_{k_2 1}\cdots a_{k_t 1}a_{k_{t+1}0}\cdots a_{t_n 0}|^2|a_{k_1 0}a_{k_2 0}\cdots a_{k_t 0}a_{k_{t+1}1}\cdots a_{t_n 1}|^2 \\ =\ & |a_{10}a_{20}\cdots a_{n0}a_{11}a_{21}\cdots a_{n1}|^2 \\ =\ & |\rho_{1,2^n}|^2 \end{aligned} \tag{30}$$

for any $\{k_1,k_2,\cdots,k_n\} = \{1,2,\cdots,n\}$. It also implies that the inequality (25) is an equality for fully separable n-qubit pure states.

(26) follows immediately from

$$|\rho_{2^i+1,2^j+1}| = \sqrt{\rho_{1,1}\rho_{2^i+2^j+1,2^i+2^j+1}}. \tag{31}$$

Next we show that the inequality (25) is also right for fully separable mixed states.

Suppose that $\rho = \sum\limits_i p_i\rho^{(i)}$, where $\rho^{(i)}$ is fully separable n-qubit pure state. Then

$$|\rho_{1,2^n}| = |\sum_i p_i\rho^{(i)}_{1,2^n}| \leq \sum_i p_i|\rho^{(i)}_{1,2^n}| = \sum_i p_i(\rho^{(i)}_{2,2}\rho^{(i)}_{3,3}\cdots\rho^{(i)}_{2^n-1,2^n-1})^{\frac{1}{2^n-2}}. \tag{32}$$

Continuously using the Hölder inequality

$$\sum_{k=1}^m |x_k y_k| \leq (\sum_{k=1}^m |x_k|^p)^{\frac{1}{p}}(\sum_{k=1}^m |y_k|^q)^{\frac{1}{q}}\ (p,q>1, \frac{1}{p}+\frac{1}{q}=1), \tag{33}$$

we get

$$
\begin{aligned}
& \sum_i p_i(\rho^{(i)}_{2,2}\rho^{(i)}_{3,3}\cdots\rho^{(i)}_{2^n-1,2^n-1})^{\frac{1}{2^n-2}} \\
= & \sum_i (p_i\rho^{(i)}_{2,2})^{\frac{1}{2^n-2}}(p_i\rho^{(i)}_{3,3}\cdots p_i\rho^{(i)}_{2^n-1,2^n-1})^{\frac{1}{2^n-2}} \\
\leq & \left(\sum_i p_i\rho^{(i)}_{2,2}\right)^{\frac{1}{2^n-2}}\left[\sum_i (p_i\rho^{(i)}_{3,3}\cdots p_i\rho^{(i)}_{2^n-1,2^n-1})^{\frac{1}{2^n-3}}\right]^{\frac{2^n-3}{2^n-2}} \\
\leq & \left(\sum_i p_i\rho^{(i)}_{2,2}\right)^{\frac{1}{2^n-2}}\left[\left(\sum_i p_i\rho^{(i)}_{3,3}\right)^{\frac{1}{2^n-3}}\left(\sum_i (\ p_i\rho^{(i)}_{4,4}\cdots p_i\rho^{(i)}_{2^n-1,2^n-1})^{\frac{1}{2^n-4}}\right)^{\frac{2^n-4}{2^n-3}}\right]^{\frac{2^n-3}{2^n-2}} \\
= & \left(\sum_i p_i\rho^{(i)}_{2,2}\right)^{\frac{1}{2^n-2}}\left(\sum_i p_i\rho^{(i)}_{3,3}\right)^{\frac{1}{2^n-2}}\left[\sum_i (\ p_i\rho^{(i)}_{4,4}\cdots p_i\rho^{(i)}_{2^n-1,2^n-1})^{\frac{1}{2^n-4}}\right]^{\frac{2^n-4}{2^n-2}} \\
\leq & \left(\sum_i p_i\rho^{(i)}_{2,2}\right)^{\frac{1}{2^n-2}}\left(\sum_i p_i\rho^{(i)}_{3,3}\right)^{\frac{1}{2^n-2}}\cdots\left(\sum_i p_i\rho^{(i)}_{2^n-1,2^n-1}\right)^{\frac{1}{2^n-2}} \\
= & (\rho_{2,2}\rho_{3,3}\cdots\rho_{2^n-1,2^n-1})^{\frac{1}{2^n-2}},
\end{aligned}
\tag{34}
$$

as claimed.

Simple algebra and the Cauchy inequality show that (26) holds for fully separable n-partite mixed states.

Observation 4 (i) and (iii) in [11] are the case $n=3$ of this theorem.

For the well-studied n-qubit GHZ states mixed with white noise, Theorem 4 constitutes a necessary and sufficient criterion for fully separable.

Theorem 5 For $\rho(p)=(1-p)|\mathrm{GHZ}_n\rangle\langle\mathrm{GHZ}_n|+\dfrac{p}{2^n}\mathrm{I}$, $\rho(p)$ is fully separable iff the entries of $\rho(p)$ satisfy the inequality (25).

Proof. Necessity is immediate from Theorem 4. Conversely if the inequality (25) holds for $\rho(p)$, i.e. $|\rho(p)_{1,2^n}|\leq(\rho(p)_{2,2}\rho(p)_{3,3}\rho(p)_{4,4}\cdots\rho(p)_{2^n-1,2^n-1})^{\frac{1}{2^n-2}}$, then there is $\frac{1-p}{2}\leq\left[(\frac{p}{2^n})^{2^n-2}\right]^{\frac{1}{2^n-2}}$, which implies that $p\geq 1-\frac{1}{2^{n-1}+1}$. Therefore, $\rho(p)$ is fully separable [15].

Observation 4 (iv) in [11] is the case $n=3$ of this theorem.

Furthermore, for high dimension and n-partite, using the Hölder inequality, we can infer:

Theorem 6 For any n-particle density matrix ρ (particle k is d_k level, $1\leq k\leq n$), if ρ is fully separable, then

$$
|\rho_{1,d_1d_2\cdots d_n}|\leq(\prod_{i\in A}\rho_{ii})^{\frac{1}{2^n-2}}, \tag{35}
$$

where A is the set of 2^n-2 numbers $\sum_{k=1}^{n-1}i_kd_{k+1}d_{k+2}\cdots d_n+i_n+1$ such that $i_k\in\{0,d_k-1\}$, and $(i_1,i_2,\cdots,i_n)\neq(0,0,\cdots,0),(d_1-1,d_2-1,\cdots,d_n-1)$, i.e., $A=\{i=\sum_{k=1}^{n-1}i_kd_{k+1}d_{k+2}\cdots d_n+i_n+1\mid i_k=0,d_k-1,\ k=1,2,\cdots,n,\ i\neq 1, i\neq d_1d_2\cdots d_n\}$.

If ρ is a fully separable n-particle pure state, then the inequality (35) is an equality.

Proof. Suppose that $\rho=|\psi\rangle\langle\psi|$ is fully separable pure state, where

$$
\begin{aligned}
|\psi\rangle = & (\sum_{i_1=0}^{d_1-1}a_{1i_1}|i_1\rangle)\otimes(\sum_{i_2=0}^{d_2-1}a_{1i_2}|i_2\rangle)\otimes\cdots\otimes(\sum_{i_n=0}^{d_n-1}a_{1i_n}|i_n\rangle) \\
= & \sum_{i_1=0}^{d_1-1}\sum_{i_2=0}^{d_2-1}\cdots\sum_{i_n=0}^{d_n-1}a_{1i_1}a_{2i_2}\cdots a_{ni_n}|i_1i_2\cdots i_n\rangle.
\end{aligned}
\tag{36}
$$

Then the elements of ρ

$$
\rho_{i,j}=a_{1i_1}a_{2i_2}\cdots a_{ni_n}a^*_{1j_1}a^*_{2j_2}\cdots a^*_{nj_n}, \tag{37}
$$

where $i=\sum_{k=1}^{n-1}i_kd_{k+1}d_{k+2}\cdots d_n+i_n+1$, $j=\sum_{k=1}^{n-1}j_kd_{k+1}d_{k+2}\cdots d_n+j_n+1$.

Since

$$
\begin{aligned}
& \rho_{\sum_{l=1}^{t}(d_{k_l}-1)d_{k_l+1}\cdots d_nd_{n+1}+1,\sum_{l=1}^{t}(d_{k_l}-1)d_{k_l+1}\cdots d_nd_{n+1}+1}\,\rho_{\sum_{l=t+1}^{n}(d_{k_l}-1)d_{k_l+1}\cdots d_nd_{n+1}+1,\sum_{l=t+1}^{n}(d_{k_l}-1)d_{k_l+1}\cdots d_nd_{n+1}+1} \\
= & |a_{k_1d_{k_1}-1}a_{k_2d_{k_2}-1}\cdots a_{k_td_{k_t}-1}a_{k_{t+1}0}\cdots a_{k_n0}|^2|a_{k_10}a_{k_20}\cdots a_{k_t0}a_{k_{t+1}d_{k_{t+1}}-1}\cdots a_{k_nd_{k_n}-1}|^2 \\
= & |a_{10}a_{20}\cdots a_{n0}a_{1d_1-1}a_{2d_2-1}\cdots a_{nd_n-1}|^2 \\
= & |\rho_{1,d_1d_2\cdots d_n}|^2,
\end{aligned}
\tag{38}
$$

for any $\{k_1, k_2, \cdots, k_t, k_{t+1}, \cdots, k_n\} = \{1, 2, \cdots, n\}$ and $d_{n+1} = 1$, this gives

$$\begin{aligned} & \left(|\rho_{1,d_1 d_2 \cdots d_n}|^2\right)^{2^n-2} \\ = & \prod_{\substack{\{k_1,\cdots,k_t,k_{t+1},\cdots,k_n\} \\ =\{1,2,\cdots,n\}}} \rho_{\sum\limits_{l=1}^{t}(d_{k_l}-1)d_{k_l+1}\cdots d_n d_{n+1}+1,\sum\limits_{l=1}^{t}(d_{k_l}-1)d_{k_l+1}\cdots d_n d_{n+1}+1} \rho_{\sum\limits_{l=t+1}^{n}(d_{k_l}-1)d_{k_l+1}\cdots d_n d_{n+1}+1,\sum\limits_{l=t+1}^{n}(d_{k_l}-1)d_{k_l+1}\cdots d_n d_{n+1}+1} \\ = & (\prod_{i\in A} \rho_{ii})^2. \end{aligned} \tag{39}$$

It implies that

$$|\rho_{1,d_1 d_2 \cdots d_n}| = (\prod_{i\in A} \rho_{ii})^{\frac{1}{2^n-2}}, \tag{40}$$

thus (35) holds for fully separable pure states. Here $A = \{i = \sum_{k=1}^{n-1} i_k d_{k+1} d_{k+2} \cdots d_n + i_n + 1 \mid i_k = 0, d_k - 1,\ k = 1, 2, \cdots, n,\ i \neq 1, i \neq d_1 d_2 \cdots d_n\}$.

One can also derive (40) by direct calculation.

Next we suppose that $\rho = \sum_i p_i \rho^{(i)}$ is an n-partite mixed state, where $\rho^{(i)} = |\psi^i\rangle\langle\psi^i|$ is fully separable. Using (40) for each $\rho^{(i)}$, we see

$$\begin{aligned} |\rho_{1,d_1 d_2 \cdots d_n}| &= |\sum_i p_i \rho^{(i)}_{1,d_1 d_2 \cdots d_n}| \\ &\le \sum_i p_i |\rho^{(i)}_{1,d_1 d_2 \cdots d_n}| = \sum_i p_i (\prod_{j\in A} \rho^{(i)}_{jj})^{\frac{1}{2^n-2}}. \end{aligned} \tag{41}$$

Let $m_2, m_3, \cdots, m_{2^n-1}$ be the elements in the set A. By the Hölder inequality, we obtain

$$\begin{aligned} & \sum_i p_i (\prod_{j\in A} \rho^{(i)}_{jj})^{\frac{1}{2^n-2}} \\ = & \sum_i (p_i \rho^{(i)}_{m_2,m_2})^{\frac{1}{2^n-2}} (p_i \rho^{(i)}_{m_3,m_3} \cdots p_i \rho^{(i)}_{m_{2^n-1},m_{2^n-1}})^{\frac{1}{2^n-2}} \\ \le & \left(\sum_i p_i \rho^{(i)}_{m_2,m_2}\right)^{\frac{1}{2^n-2}} \left[\sum_i (p_i \rho^{(i)}_{m_3,m_3} \cdots p_i \rho^{(i)}_{m_{2^n-1},m_{2^n-1}})^{\frac{1}{2^n-3}}\right]^{\frac{2^n-3}{2^n-2}} \\ \le & \left(\sum_i p_i \rho^{(i)}_{m_2,m_2}\right)^{\frac{1}{2^n-2}} \left(\sum_i p_i \rho^{(i)}_{m_3,m_3}\right)^{\frac{1}{2^n-2}} \left[\sum_i (p_i \rho^{(i)}_{m_4,m_4} \cdots p_i \rho^{(i)}_{m_{2^n-1},m_{2^n-1}})^{\frac{1}{2^n-4}}\right]^{\frac{2^n-4}{2^n-2}} \\ \le & \left[\sum_i (p_i \rho^{(i)}_{m_2,m_2})\right]^{\frac{1}{2^n-2}} \left[\sum_i (p_i \rho^{(i)}_{m_3,m_3})\right]^{\frac{1}{2^n-2}} \cdots \left[\sum_i (p_i \rho^{(i)}_{m_{2^n-1},m_{2^n-1}})\right]^{\frac{1}{2^n-2}} \\ = & (\rho_{m_2,m_2} \rho_{m_3,m_3} \cdots \rho_{m_{2^n-1},m_{2^n-1}})^{\frac{1}{2^n-2}} \\ = & (\prod_{i\in A} \rho_{ii})^{\frac{1}{2^n-2}}. \end{aligned} \tag{42}$$

Combining (41) and (42) gives the inequality (35), as required.

IV. CONCLUSION

We derive separability criteria for n-qubit and n-qudit quantum states directly in terms of matrix elements. Some of them are also sufficient conditions for genuine entanglement of n-partite quantum states. One of the resulting criteria is also necessary and sufficient condition for a class of n-partite states. We give clear and complete proof of each criterion from general partition by using the Cauchy inequality and Hölder inequality.

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